Phil 97 Reverse Mathematics

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1 Course Description

Mathematicians (usually) prove theorems from axioms, and their proofs show that the axioms used are *sufficient* for securing the truth of those theorems. Are the axioms used *necessary*, or would weaker principles suffice? This question is the subject of *reverse mathematics*, which is so named because it investigates whether a given theorem proves, over a weak base theory, a given set existence axiom sufficient to prove it. Results of this kind are called *reversals*, and it is an intriguing empirical fact that most theorems of undergraduate mathematics "reverse" to one of five set existence axioms. Reversals have established a rich network of connections between logic (particularly computability theory), analysis, combinatorics, and other branches of mathematics.

This course is an introduction to reverse mathematics and the philosophical questions concomitant with it. Specifically, we will see the "Big Five" subsystems of second-order arithmetic and weigh the philosophical claims made about the reverse mathematics research program: Is it really a successor to Hilbert's Program, which sought to prove the consistency of mathematical theories finitistically? What exactly do we gain epistemically from reversals? What is the significance of the Big Five subsystems, and how do they relate to traditional schools in the foundations of mathematics like finitism or predicativism?

The prerequisites for the course are (1) having taken a proof-based mathematics or logic course and (2) being open to learning technical material. It will not be assumed that one knows about e.g. the competing schools in the foundations of mathematics. Please email the instructor if you have any questions.

2 Learning Goals

By the end of this course, you will be able to

- read and write technical philosophy papers,
- present detailed material clearly,
- describe the different subsystems of second order arithmetic and what is provable (or not) in each of them, and
- explain how reverse mathematics as a research program bears on the foundations of mathematics.

3 Course Mechanics

Course Texts (available on Canvas)

(**Main Text**) Stephen G. Simpson, *Subsystems of Second Order Arithmetic.*

Denis Hirschfeldt, *Slicing the Truth.*

John Stillwell, *Reverse Mathematics: Proofs from the Inside Out.*

Another useful resource is the Reverse Mathematics Zoo.

Assignments

- Multiple low-stakes presentations of theorems and philosophical arguments, 20-30 minutes each.
- One expository paper, which will be revised once.
- One term paper of 8-10 pages, which will be revised once. Prompts will be posted on the Canvas site, but you are encouraged to explore a topic that interests you.

  The final course grade is determined by participation (33%), the revised expository paper (33%), and the revised term paper (33%).

4 Outline

Week 1 Introduction: First and (subsystems of) second order arithmetic

Weeks 2-7 The "Big Five" subsystems and mathematics within them

(2) ACA_0
Required reading: Stillwell chapters 1 and 2
Recommended for review: Simpson §1.2-3, §1.7-8, Stillwell chapters 3-5

(3) WKL_0
Required reading: Stillwell chapter 7, "On the Infinite" by David Hilbert.
Recommended for review: Stillwell §4.5, §4.7, and chapter 6

(4) WKL_0 and Hilbert's Program
Required reading: Simpson, "Partial realizations of Hilbert's Program"

(5) Π^0_2-conservativity of WKL_0 over PRA, Introduction to ATR_0

(6) Mathematics in ATR_0
Required reading: "The prehistory of the subsystems of second-order arithmetic" by Walter Dean & Sean Walsh, Simpson § I.11
Recommended: §2.4 of the Stanford Encyclopedia entry on *philosophy of mathematics*

(7) Reversals of comparability of countable wellorderings, open determinacy, and the open Ramsey theorem to ATR_0
(Student presentations.)

Weeks 8-10 Disarray

(8) Silver's theorem in Π^1_1-CA, Intro to Ramsey's theorem for pairs and the Reverse Mathematics Zoo.
Required reading: Hirschfeldt chapter 5 and §6.1, the Introduction to "Open questions in reverse mathematics" by Montalbán.
Recommended: Simpson § VI.3. Peruse the Reverse Mathematics Zoo.

(9) Ramsey's theorem for pairs and the Reverse Mathematics Zoo

(10) The reverse mathematics of determinacy.
Required reading: "Higher set theory and mathematical practice" by Harvey Friedman.

Weeks 11-12 Philosophy & Methodology

(11) The philosophy of reverse mathematics.
Required reading: "Set existence and closure conditions: unravelling the standard view of reverse mathematics" by Benedict Eastaugh.

(12) The philosophy of reverse mathematics.

5 Papers (available on Canvas)

C. T. Chong, Theodore A. Slaman, and Yue Yang, The metamathematics of stable Ramsey's theorem for pairs. *J. Amer. Math. Soc.*, 27:863-892, 2014.

Walter Dean and Sean Walsh, The prehistory of the subsystems of second-order arithmetic. *The Review of Symbolic Logic*, 10(2):357–396, 2017.

Benedict Eastaugh, Set existence and closure conditions: unravelling the standard view of reverse mathematics. *Philosophia Mathematica* 27(2):153–176, 2019

Harvey Friedman, Higher set theory and mathematical practice. *Annals of Mathematics* 2:325-357, 1971.

Harvey Friedman, Some systems of second order arithmetic and their use. *Proceedings of the International Congress of Mathematicians, Vancouver 1974* 1:235-242, 1975.

David Hilbert, On the Infinite.

Penelope Maddy, *Defending the Axioms.*

Antonio Montalbán, Open questions in reverse mathematics. *Bulletin of Symbolic Logic*, 17 (2011), 431-454.

Antonio Montalbán and Richard Shore, The limits of determinacy in second order arithmetic. *Proceedings of the London Math Society*, 104 (2012), 223-252.

Richard Shore, Reverse Mathematics: the Playground of Logic. *Bulletin of Symbolic Logic* 16 (2010), 378-402.

Stephen G. Simpson, The Gödel Hierarchy and Reverse Mathematics. *Kurt Gödel: Essays for his Centennial.*

Stephen G. Simpson, Partial realizations of Hilbert's program. *Journal of Symbolic Logic* (1988).